A NEW TYPE OF PROPAGATION

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Abstract

H. Alfven and others have shown that when a conducting fluid is placed in a magnetic field the interaction of the electromagnetic and hydrodynamic forces generates a new type of wave, the nagneto-hydrodynamic wave. Using a description of these waves in an ideal fluid as a prototype, we briefly discuss their properties for various types of conducting fluids. A list of references is given.

Introduction

When a conducting liquid in hydrodynamic motion is exposed to a magnetic field, it becomes electrically polarized. This polarization induces in the liquid an electrical conduction current, and the interaction of this current with the nagnetic field produces a mechanical force which modifies the hydrodynamic motion. To examine the interaction mathematically, it is necessary to solve simultaneously the macroscopic Maxwell equations and the equation of motion. In general, the problem is non-linear. However, for small signals the non-linearity disappears and the problem amounts to solving a wave equation. Thus, at least in this restricted case, the interaction gives rise to a wave phenomenon which after H. Alfvén is called a magneto-hydrodynamic wave.

If the medium is perfectly conducting, non-viscous, and incompressible a plane magneto-hydrodynamic wave will travel without amplification or attenuation in the direction of the applied magnetic field with a velocity proportional to the magnetic field and inversely proportional to the square root of the density. The wave will consist of a TEM wave, a transverse sound wave, and a pressure wave, all traveling with the same velocity. The TEM wave behaves like an ordinary TEM wave traveling through a medium of high dielectric constant. Moreover, the existence of a transverse sound wave implies that the applied magnetic field gives the non-viscous medium on effective viscosity. If the conductivity of the medium is very high but not infinite, the waves are exponentially damped. And if the conductivity is negative, the waves are amplified.

If the conducting medium is compressible, a longitudinal sound wave can propagate through it and in so doing will interact with the magnetic field to produce a combination wave motion consisting of a longitudinal sound wave and a magneto-hydrodynamic wave. The equation of motion is like the one for sound except that an electromagnetic force term is added to the compression force. This equation possesses three solutions or modes.

Whether the medium is a conducting liquid or an ionized gas (for a plasma magneto-hydrodynamic waves are present when the frequency is much less than the gyro-frequency of the ions) the mechanism of interaction between the medium and the electromagnetic field is essentially the same: the magnetic lines of force follow the motion of the fluid and consequently are stretched or contracted depending upon the nature of the fluid motion. Since the magnetic energy is increased when the magnetic lines are stretched and, conversely, decreased when contracted, an interchange of mechanical energy and magnetic energy can take place. In fact, it has been shown by Batchelor that a weak magnetic field in a turbulent, conducting liquid will exponentially grow until the magnetic energy is equal to the energy contained in the small-scale components of the turbulence.

The special applications of magneto-hydrodynamic theory to electronic devices and ionospheric propagation will be reported in detail elsewhere. The sole purpose of this note is to invite the attention of radio engineers to this interesting, new type of propagation. A list of references is provided for those who would like to pursue the theory further.

Ideal Medium

For a conducting liquid magneto-hydrodynamic theory is based upon the following equations:

$$\nabla \times \underline{E} = -\frac{\partial}{\partial t}\underline{B} \tag{1}$$

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial}{\partial t}\underline{D} \tag{2}$$

$$\underline{J} = \sigma(\underline{E} + \underline{v} \times \underline{B}) \tag{3}$$

$$\rho\frac{d}{dt}\underline{v} = \underline{J} \times \underline{B} - \nabla p \tag{4}$$

where ρ is the density of the liquid, σ is its conductivity, $\underline{v}(\underline{r},t)$ is the velocity of the liquid at point $\underline{r}$ and time t, p is the pressure, and all other symbols have their customary meanings. (1) and (2) are the macroscopic Maxwell equations, (3) is Ohm's law for medium moving with velocity $\underline{v}$, and (4) is the equation of motion. The liquid is homogeneous and isotropic, and, therefore, is independent of $\underline{r}$ and t and is a scalar quantity. The time derivative of the velocity, $\frac{d}{dt}\underline{V}$, is taken along trajectory and is explicitly equal to $\frac{\partial}{\partial t}\underline{V} + (\underline{V}\cdot\nabla)\underline{V}$.

The coupling of electromagnetic and hydrodynamic quantities is through the $\underline{v} \times \underline{B}$ polarization term in (3) and the $\underline{J} \times \underline{B}$ force term in (4). When $\sigma = 0$ the coupling is zero and the electromagnetic and hydrodynamic phenomena are independent. When $\sigma \to \infty$ and, hence, $(\underline{E} + \underline{v} \times \underline{B}) \to 0$, the coupling is tight. In other words the degree of coupling depends on the conductivity of the liquid. In the equation of motion (4) we have left out the forces of gravitation and viscosity for we have assumed that the liquid is non-viscous and the gravitational field has only a negligible effect upon the phenomenon implied by the equations.

If we further assume that the liquid is incompressible, i.e.

$$\nabla \cdot \underline{v} = 0, \tag{5}$$

and that the liquid is immersed in a homogeneous, static, magnetic field, $\underline{H}o$, the equations (1) to (5) lead to the conclusion that a wave is propagated in the direction of $\underline{H}o$ with a velocity, $\underline{V}$, where

$$\underline{V} = \pm \underline{H}o\sqrt{\mu/\rho} \tag{6}$$

To show that (1) - (5) yield a wave equation we first neglect the displacement current $\frac{\partial}{\partial t}\underline{D}$ in (2), as we do in the theory of skin effect. Then we let

$$\underline{H} = \underline{H}o + \underline{H}' \tag{7}$$

where $\underline{H}o$, the applied homogeneous static field, is independent of space and time and H' is the magnetic field produced by the induced current density $\underline{J}$.

Taking the curl of (2), neglecting the displacement current, and using (1) and (3) we get

$$\nabla \times \nabla \times \underline{H}' = -\mu\sigma \frac{\partial}{\partial t}\underline{H}' + \mu\sigma\, \nabla \times (\underline{v} \times \underline{H}) \tag{8}$$

From vector analysis,

$$\nabla \times (\underline{v} \times \underline{H}) = \nabla \times (\underline{v} \times \underline{H}') + (\underline{H}o \cdot \nabla)\, \underline{v} \tag{9}$$

where we have used $\nabla \cdot \underline{v} = 0$, the incompressibility condition. Now if we choose a cartesian coordinate system with z-axis parallel to $\underline{H}o$, $\underline{H}o \cdot \nabla$ becomes Ho $\frac{\partial}{\partial z}$. Substituting (9) into (8) we obtain

$$\nabla \times \nabla \times \underline{H}' = -\mu\sigma \frac{\partial}{\partial t}\underline{H}' + \mu\sigma\, \nabla \times (\underline{v} \times \underline{H}') + Ho \frac{\partial}{\partial z}\underline{v}.$$

For small signals $\underline{v}$ and $\underline{H}'$ are small quantities and $\underline{v} \times \underline{H}'$ is the square of a small quantity and hence negligible. With this restriction to small amplitudes, the above equation becomes

$$\left(\frac{\partial}{\partial t}\underline{H}' + \frac{1}{\mu\sigma}\nabla \times \nabla \times \underline{H}'\right) = Ho \frac{\partial}{\partial z}\underline{v} \tag{10}$$

Now we focus our attention on the equation of motion (4). Since we have neglected the displacement current, $\nabla \times \underline{H} = \underline{J}$ and (4) becomes

$$\rho \frac{d}{dt}\underline{v} = \mu\, (\nabla \times \underline{H}) \times \underline{H} - \nabla p.$$

Since $\nabla \times \underline{H} = \nabla \times \underline{H}'$ and $\frac{d}{dt}\underline{v} = \frac{\partial}{\partial t}\underline{v} + (\underline{v} \cdot \nabla)\,\underline{v}$, we have

$$\rho\left(\frac{\partial}{\partial t}\underline{v} + (\underline{v}\cdot\nabla)\underline{v}\right) = \mu(\nabla\times\underline{H}')\times\underline{H} - \nabla p \qquad (11)$$

The term $(\underline{v}\cdot\nabla)\underline{v}$ is of the second order and we neglect it. Moreover,

$$(\nabla\times\underline{H}')\times\underline{H} = (\nabla\times\underline{H}')\times\underline{Ho} + (\nabla\times\underline{H}')\times\underline{H}'$$

and we neglect $(\nabla\times\underline{H}')\times\underline{H}'$ for the same reason. Consequently, by a little vector analysis

$$(\nabla\times\underline{H}')\times\underline{H} = (\nabla\times\underline{H}')\times\underline{Ho} = -\nabla(\underline{Ho}\cdot\underline{H}') + (\underline{Ho}\cdot\nabla)\underline{H}' \qquad (12)$$

Recalling that $\underline{Ho}\cdot\nabla = Ho\frac{\partial}{\partial z}$, (11) and (12) yield

$$\rho\frac{\partial}{\partial t}\underline{v} = -\nabla(p + \mu\,\underline{Ho}\cdot\underline{H}') + \mu\,Ho\frac{\partial}{\partial z}\underline{H}' \qquad (13)$$

Let us now take the curl of (10) and (13). Since $\nabla\times\underline{H}' = \underline{J}$, the curl of (10) gives

$$\frac{\partial}{\partial t}\underline{J} + \frac{1}{u\sigma}\nabla\times\nabla\times\underline{J} = Ho\frac{\partial}{\partial z}(\nabla\times\underline{v}) \qquad (14)$$

where $\nabla\times\underline{v}$ is the known as the vorticity vector. And the curl of (13) yields

$$\rho\frac{\partial}{\partial t}(\nabla\times\underline{v}) = \mu\,Ho\frac{\partial}{\partial z}\nabla\times\underline{H}' = \mu\,Ho\frac{\partial}{\partial z}\underline{J} \qquad (15)$$

Multiplying (14) by $\frac{1}{Ho}\frac{\partial}{\partial t}$ and (15) by $\frac{1}{\rho}\frac{\partial}{\partial z}$ and then substracting we obtain an equation in $\underline{J}$:

$$\frac{\partial^2}{\partial t^2}\underline{J} - \frac{\mu Ho^2}{\rho}\frac{\partial^2}{\partial z^2}\underline{J} + \frac{1}{\mu\sigma}\frac{\partial}{\partial t}\nabla\times\nabla\times\underline{J} = 0 \qquad (16)$$

When σ is large so that the third term is negligible, (16) becomes the wave equation,

$$\left(\frac{\partial^2}{\partial t^2} - V^2\frac{\partial^2}{\partial z^2}\right)\underline{J} = 0 \qquad (17)$$

where $V = \sqrt{\frac{\mu Ho^2}{\rho}}$ is the speed of the current wave.

If σ is small so that the second term of (16) is neglibigle, we get the skin effect equation,

$$\frac{\partial}{\partial t}\underline{J} + \frac{1}{\mu\sigma}\nabla\times\nabla\times\underline{J} = 0 \qquad (18)$$

We now assume that the conductivity of the liquid is infinite and therefore $\underline{J}$ must satisfy (17). If $\underline{J}$ has only an x-component and is a function of z and t only, we see that

$$J_x \sim \cos w\,(t \pm z/V) \qquad (19)$$

where A is a constant and w is the angular frequency. The upper and lower signs correspond to waves traveling along the negative and positive z-axis. Similarly $\underline{E}$, $\underline{H}'$, and $\underline{v}$ are functions of z and t only; $\underline{H}'$ and $\underline{v}$ have only y-components, $\underline{E}$ has only an x-component. And H'_y, v_y, E_x must satisfy the wave equation (17). A consistent set of plane wave solutions are the following:

$$H'_y = A \sin w\,(t - z/V) \tag{20}$$

$$J_x = \frac{w}{V} A \cos w\,(t - z/V) \tag{21}$$

$$E_x = \mu VA \sin w\,(t - z/V) \tag{22}$$

$$V_y = -VA \sin w\,(t - z/V) \tag{23}$$

E_x and H'_y are the electric and magnetic components of a TEM wave traveling along the positive z-axis. The magnetic lines of force move with same velocity vy as the fluid and therefore can be considered as "frozen" in this perfectly conducting fluid. Moreover, we see from (23) that the medium supports a transverse sound wave even though we assumed that the liquid was non-viscous.

Other Media

If the medium is compressible and hence can support a longitudinal sound wave of velocity W, there distinct modes of plane wave propagation are realizable. We denote the displacement of the medium by $\underline{\xi}$, the wave normal by $\underline{n}$, and the angle between them by φ. In the first mode $\underline{\xi}$ is perpendicular to $\underline{n}$ and $\underline{H}$ is directed along the planes of constant phase and we get a magneto-hydrodynamic wave identical to the one in incompressible medium traveling with velocity $V \cos\varphi$. In the second and third modes $\underline{\xi}$, $\underline{n}$, and $\underline{H}$ are coplanar, and if $V \ll W$ the velocities of propagation are respectively $V \cos\varphi$ and $(W^2 + v^2 \cos^2)^{1/2}$. The third mode like sound is isotropic whereas the first two are strongly anisotropic.

If the medium is an ionized gas the theory is based upon (1) and (2) in conjunction with Lorentz definition of current,

$$\underline{J} = \sum_n N_n e_n \underline{v}_n$$

and the equation of motion,

$$m \frac{\partial}{\partial t} \underline{v} = e\,(\underline{E} + \underline{v} \times \underline{B})$$

where N_n is the number of particles of the n-th type, e_n the charge, and $\underline{v}_n$ the velocity, and the summation is over all particles. The dielectric constant turns out to be a dyadic. For frequencies well below the gyrofrequencies of the ions, the wave motion becomes magneto-hydrodynamic.

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